VII. *Researches on the Partition of Numbers.* *By* Arthur Cayley, *Esq.*

Received April 14,—Read May 3 and 10, 1855.

I PROPOSE to discuss the following problem: "To find in how many ways a number q can be made up of the elements $a, b, c, ..$ each element being repeatable an indefinite number of times." The required number of partitions is represented by the notation

$$P(a, b, c, ..)q,$$

and we have, as is well known,

$$P(a, b, c, ..)q = \text{coefficient } x^q \text{ in } \frac{1}{(1-x^a)(1-x^b)(1-x^c)..},$$

where the expansion is to be effected in ascending powers of x.

It may be as well to remark that each element is to be considered as a separate and distinct element, notwithstanding any equalities which may exist between the numbers $a, b, c, ..$; thus, although $a=b$, yet $a+a+a+$ &c. and $a+a+b+$ &c. are to be considered as two different partitions of the number q, and so in all similar cases.

The solution of the problem is thus seen to depend upon the theory, to which I now proceed, of the expansion of algebraical fractions.

Consider an algebraical fraction $\frac{\phi x}{fx}$,

where the denominator is the product of any number of factors (the same or different) of the form $1-x^m$. Suppose in general that $[1-x^m]$ denotes the irreducible factor of $1-x^m$, *i. e.* the factor which, equated to zero, gives the prime roots of the equation $1-x^m=0$. We have

$$1-x^m = \Pi[1-x^{m'}],$$

where m' denotes any divisor whatever of m (unity and the number m itself not excluded). Hence, if a represent a divisor of one or more of the indices m, and k be the number of the indices of which a is a divisor, we have

$$fx = \Pi[1-x^a]^k.$$

Now considering apart from the others one of the multiple factors $[1-x^a]^k$, we may write $fx = [1-x^a]^k f_{\prime}x$.

Suppose that the fraction $\frac{\phi x}{fx}$ is decomposed into simpler fractions, in the form

$$\begin{aligned}\frac{\phi x}{fx} &= \mathrm{I}(x) \\ &+ (x\partial_x)^{k-1}\frac{\theta x}{[1-x^a]} + (x\partial_x)^{k-2}\frac{\theta_1 x}{[1-x^a]} \cdot\cdot + \frac{\theta_{k-1}x}{[1-x^a]} \\ &+ \&\text{c.},\end{aligned}$$

where $\mathrm{I}(x)$ denotes the integral part, and the &c. refers to the fractional terms depending upon the other multiple factors, such as $[1-x^a]^k$. The functions θx are to be considered as functions with indeterminate coefficients, the degree of each such function being inferior by unity to that of the corresponding denominator; and it is proper to remark that the number of the indeterminate coefficients in all the functions θx together is equal to the degree of the denominator fx.

The term $(x\partial_x)^{k-1}\dfrac{\theta x}{[1-x^a]}$ may be reduced to the form

$$\frac{gx}{[1-x^a]^k}+\frac{g_1x}{[1-x^a]^{k-1}}+\&\text{c.},$$

the functions gx being of the same degree as θx, and the coefficients of these functions being linearly connected with those of the function θx. The first of the foregoing terms is the only term on the right-hand side which contains the denominator $[1-x^a]^k$; hence, multiplying by this denominator and then writing $[1-x^a]=0$, we find

$$\frac{\phi x}{f_1 x}=gx,$$

which is true when x is any root whatever of the equation $[1-x^a]=0$. Now by means of the equation $[1-x^a]=0$, $\dfrac{\phi x}{f_1x}$ may be expressed in the form of a rational and integral function $\mathrm{G}x$, the degree of which is less by unity than that of $[1-x^a]$. We have therefore $\mathrm{G}x=gx$, an equation which is satisfied by each root of $[1-x^a]=0$, and which is therefore an identical equation; gx is thus determined, and the coefficients of θx being linear functions of those of gx, the function θx may be considered as determined. And this being so, the function

$$\frac{\phi x}{fx}-(x\partial_x)^{k-1}\frac{\theta x}{[1-x^a]}$$

will be a fraction the denominator of which does not contain any power of $[1-x^a]$ higher than $[1-x^a]^{k-1}$; and therefore $\theta_1 x$ can be found in the same way as θx, and similarly $\theta_2 x$, and so on. And the fractional parts being determined, the integral part may be found by subtracting from $\dfrac{\phi x}{fx}$ the sum of the fractional parts, so that the fraction $\dfrac{\phi x}{fx}$ can by a direct process be decomposed in the above-mentioned form.

Particular terms in the decomposition of certain fractions may be obtained with great facility. Thus m being a prime number, assume

$$\frac{1}{(1-x^2)(1-x^3)\,..\,(1-x^m)}=\&\text{c.}+\frac{\theta x}{[1-x^m]};$$

then observing that $(1-x^m)=(1-x)[1-x^m]$, we have for $[1-x^m]=0$,

$$\theta x=\frac{1}{(1-x)(1-x^2)\,..\,(1-x^{m-1})}.$$

Now u being any quantity whatever and x being a root of $[1-x^m]=0$, we have identically

$$[1-u^m]=(u-x)(u-x^2)..(u-x^{m-1});$$

and therefore putting $u=1$, we have $\quad m=(1-x)(1-x^2)..(1-x^{m-1})$, and therefore

$$\theta x=\frac{1}{m},$$

whence

$$\frac{1}{(1-x^2)(1-x^3)..(1-x^m)}=\&\text{c.}+\frac{1}{m}\,\frac{1}{[1-x^m]}.$$

Again, m being as before a prime number, assume

$$\frac{1}{(1-x)(1-x^2)..(1-x^m)}=\&\text{c.}+\frac{\theta x}{[1-x^m]},$$

we have in this case for $[1-x^m]=0$,

$$\theta x=\frac{1}{(1-x)^2(1-x^2)..(1-x^{m-1})},$$

which is immediately reduced to $\theta x=\frac{1}{m}\,\frac{1}{1-x}$. Now

$$\frac{[1-u^m]}{u-x}=\frac{[1-u^m]-[1-x^m]}{u-x}=(1+u+..+u^{m-2})+(1+u..+u^{m-3})x..+(1+u)x^{m-3}+x^{m-2};$$

or putting $u=1$,

$$\frac{m}{1-x}=\overline{m-1}+\overline{m-2}x..+x^{m-2};$$

and substituting this in the value of θx, we find

$$\frac{1}{(1-x)(1-x^2)..(1-x^m)}=\&\text{c.}+\frac{1}{m^2}\,\frac{(m-1)+(m-2)x..+x^{m-2}}{[1-x^m]}.$$

The preceding decomposition of the fraction $\frac{\phi x}{fx}$ gives very readily the expansion of the fraction in ascending powers of x. For, consider a fraction such as

$$\frac{\theta x}{[1-x^a]},$$

where the degree of the numerator is in general less by unity than that of the denominator; we have

$$1-x^a=[1-x^a]\Pi[1-x^{a'}],$$

where a' denotes any divisor of a (including unity, but not including the number a itself). The fraction may therefore be written under the form

$$\frac{\theta x\Pi[1-x^{a'}]}{1-x^a},$$

where the degree of the numerator is in general less by unity than that of the denominator, *i. e.* is equal to $\overline{a-1}$. Suppose that b is any divisor of a (including unity, but not including the number a itself), then $1-x^b$ is a divisor of $\Pi[1-x^{a'}]$, and therefore of the numerator of the fraction. Hence representing this numerator by

$$A_0+A_1x\ldots+A_{a-1}x^{a-1},$$

and putting $a=bc$, we have (corresponding to the case $b=1$)

$$A_0+A_1+A_2\ldots+A_{a-1}=0,$$

and generally for the divisor b,

$$A_0+A_b\ldots+A_{(c-1)b}=0$$
$$A_1+A_{b+1}\ldots+A_{(c-1)b+1}=0$$
$$\cdot$$
$$\cdot$$
$$A_{b-1}+A_{2b-1}\ldots+A_{cb-1}=0.$$

Suppose now that a_q denotes a circulating element to the period a, *i. e.* write

$$a_q=1 \quad q=0 \text{ (mod. } a)$$
$$a_q=0 \text{ in every other case.}$$

A function such as

$$A_0a_q+A_1a_{q-1}\ldots+A_{a-1}a_{q-a+1}$$

will be a circulating function, or circulator to the period a, and may be represented by the notation

$$(A_0, A_1, \ldots A_{a-1}) \text{ circlor } a_q.$$

In the case however where the coefficients A satisfy, for each divisor b of the number a, the above-mentioned equations, the circulating function is what I call a prime circulator, and I represent it by the notation

$$(A_0, A_1, \ldots A_{a-1}) \text{ pcr } a_q.$$

By means of this notation we have at once

$$\text{coefficient } x_q \text{ in } \frac{\theta x}{[1-x^a]}=(A_0, A_1\ldots A_{a-1}) \text{ pcr } a_q,$$

and thence also

$$\text{coefficient } x_q \text{ in } (x\partial_x)^r\frac{\theta x}{[1-x^a]}=q^r(A_0, A_1\ldots A_{a-1}) \text{ pcr } a_q.$$

Hence assuming that in the fraction $\frac{\phi x}{fx}$ the degree of the numerator is less than that of the denominator (so that there is not any integral part), we have

$$\text{coefficient } x_q \text{ in } \frac{\phi x}{fx}=\Sigma\ q^r(A_0, A_1, \ldots A_{a-1}) \text{ pcr } a_q;$$

or, if we wish to put in evidence the non-circulating part arising from the divisor $a=1$,

$$\text{coefficient } x_q \text{ in } \frac{\phi x}{fx}=Aq^{k-1}+Bq^{k-2}\ldots+Lq+M$$
$$+\Sigma\ q^r(A_0, A_1\ldots A_{a-1}) \text{ pcr } a_q;$$

where k denotes the number of the factors $1-x^m$ in the denominator fx, a is any divisor (unity excluded) of one or more of the indices m; and for each value of a r extends from $r=0$ to $r=k-1$, where k denotes the number of indices m of which

a is a divisor. The particular results previously obtained show, that m being a prime number,

$$\text{coefficient } x^q \text{ in } \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^m)} = \&\text{c.} + \frac{1}{m}(1, -1, 0, 0, \ldots) \text{ pcr } m_q,$$

and

$$\text{coefficient } x^q \text{ in } \frac{1}{(1-x)(1-x^2)\ldots(1-x^m)} = \&\text{c.} + \frac{1}{m^2}(m-1, -1, -1, \ldots) \text{ pcr } m_q.$$

Suppose, as before, that the degree of ϕx is less than that of fx, and let the analytical expression above obtained for the coefficient of x^q in the expansion in ascending powers of x of the fraction $\frac{\phi x}{fx}$ be represented by Fq, it is very remarkable that if we expand $\frac{\phi x}{fx}$ in descending powers of x, then the coefficient of x^q in this new expansion (q is here of course negative, since the expansion contains only negative powers of x) is precisely equal to $-\mathrm{F}q$; this is in fact at once seen to be the case with respect to each of the partial fractions into which $\frac{\phi x}{fx}$ has been decomposed, and it is consequently the case with respect to the fraction itself*. This gives rise to a result of some importance. Suppose that ϕx and fx are respectively of the degrees N and D; it is clear from the form of fx that we have $f\left(\frac{1}{x}\right) = (-)^{\mathrm{D}} x^{-\mathrm{D}} fx$; and I suppose that ϕx is also such that $\phi\left(\frac{1}{x}\right) = (\pm)^{\mathrm{N}} x^{-\mathrm{N}} \phi x$; then writing $\mathrm{D}-\mathrm{N}=h$, and supposing that $\frac{\phi x}{fx}$ is expanded in descending powers of x, so that the coefficient of x^q in the expansion is $-\mathrm{F}q$, it is in the first place clear that the expansion will commence with the term x^{-h}, and we must therefore have

$$\mathrm{F}q = 0$$

for all values of q from $q=-1$ to $q=-(h-1)$.

Consider next the coefficient of a term x^{-h-q}, where q is 0 or positive; the coefficient in question, the value of which is $-\mathrm{F}(-h-q)$, is obviously equal to the coefficient of x^{h+q} in the expansion in ascending powers of x of $\dfrac{\phi\frac{1}{x}}{f\left(\frac{1}{x}\right)}$, *i. e.* to

$$(\pm)^{\mathrm{N}}(-)^{\mathrm{D}} \text{ coefficient } x^{h+q} \text{ in } \frac{x^h \phi x}{fx},$$

or what is the same thing, to

$$(\pm)^{\mathrm{N}}(-)^{\mathrm{D}} \text{ coefficient } x^q \text{ in } \frac{\phi x}{fx};$$

and we have therefore, q being zero or positive,

$$\mathrm{F}(-h-q) = -(\pm)^{\mathrm{N}}(-)^{\mathrm{D}} \mathrm{F}q.$$

In particular, when $\phi x = 1$, $\quad \mathrm{F}q = 0$

* The property is a fundamental one in the general theory of developments.

for all values of q from $q=-1$ to $q=-(D-1)$; and q being 0 or positive,

$$F(-D-q)=(-)^{D-1}Fq.$$

The preceding investigations show the general form of the function $P(a, b, c, \ldots)q$, viz. that

$$P(a, b, c, ..)q=Aq^{k-1}+Bq^{k-2}..+Lq+M+\Sigma\, q^r(A_0, A_1, .. A_{l-1}) \text{ pcr } l_q,$$

a formula in which k denotes the number of the elements $a, b, c,$ .. &c., and l is any divisor (unity excluded) of one or more of these elements; the summation in the case of each such divisor extends from $r=0$ to $r=k-1$, where k is the number of the elements $a, b, c, \ldots$ &c. of which l is a divisor; and the investigations indicate how the values of the coefficients A of the prime circulators are to be obtained. It has been moreover in effect shown, that if $D=a+b+c+..$, then, writing for shortness $P(q)$ instead of $P(a, b, c, ..)q$, we have

$$P(q)=0$$

for all values of q from $q=-1$ to $q=-(D-1)$, and that q being 0 or positive,

$$P(-D-q)=(-)^{D-1}P(q);$$

these last theorems are however uninterpretable in the theory of partitions, and they apply only to the analytical expression for $P(q)$.

I have calculated the following particular results :—

$$P(1, 2)q \quad =\frac{1}{4}\left\{2q+3 \atop +(1, -1) \text{ pcr } 2_q\right\}$$

$$P(1, 2, 3)q \quad =\frac{1}{72}\left\{\begin{array}{l}6q^2+36q+47\\ +9(1, -1) \text{ pcr } 2_q\\ +8(2, -1, -1) \text{ pcr } 3_q\end{array}\right\}$$

$$P(1, 2, 3, 4)q =\frac{1}{288}\left\{\begin{array}{l}2q^3+30q^2+135q+175\\ +(9q+45)(1, -1) \text{ pcr } 2_q\\ +32\ (1, 0, -1) \text{ pcr } 3_q\\ +36\ (1, 0, -1, 0) \text{ pcr } 4_q\end{array}\right\}$$

$$P(1, 2, 3, 4, 5)q=\frac{1}{86400}\left\{\begin{array}{l}30q^4+900q^3+9300q^2+38250q+50651\\ +(1350q+10125)\ (1, -1) \text{ pcr } 2_q\\ +3200\ (2, -1, -1) \text{ pcr } 3_q\\ +5400\ (1, 1, -1, -1) \text{ pcr } 4_q\\ +3456(4, -1, -1, -1, -1) \text{ pcr } 5_q\end{array}\right\}$$

$$P(2)q \quad =\frac{1}{2}\left\{1 \atop +(1, -1) \text{ pcr } 2_q\right\}$$

$$P(2,3)q = \frac{1}{12}\left\{\begin{array}{l} 2q+5 \\ +3\ (1,-1) \text{ pcr } 2_q \\ +4(1,-1,0) \text{ pcr } 3_q \end{array}\right\}$$

$$P(2,3,4)q = \frac{1}{288}\left\{\begin{array}{l} 6q^2+54q+107 \\ +(18q+81)(1,-1) \text{ pcr } 2_q \\ +32\ (2,-1,-1) \text{ pcr } 3_q \\ +36\ (1,-1,-1,1) \text{ pcr } 4_q \end{array}\right\}$$

$$P(2,3,4,5)q = \frac{1}{1440}\left\{\begin{array}{l} 2q^3+42q^2+267q+497 \\ +(45q+315)(1,-1) \text{ pcr } 2_q \\ +160\ (1,-1,0) \text{ pcr } 3_q \\ +180\ (1,0,-1,0) \text{ pcr } 4_q \\ +288\ (1,-1,0,0,0) \text{ pcr } 5_q \end{array}\right\}$$

$$P(2,3,4,5,6)q = \frac{1}{172800}\left\{\begin{array}{l} 10q^4+400q^3+5550q^2+31000q+56877 \\ +(450q^2+9000q+39075)(1,-1) \text{ pcr } 2_q \\ +3200q\ (1,-1,0) \text{ pcr } 3_q \\ +1600\ (21,-19,-2) \text{ pcr } 3_q \\ +10800\ (1,0,-1,0) \text{ pcr } 4_q \\ +6912\ (4,-1,-1,-1,-1) \text{ pcr } 5_q \\ +4800\ (1,-1,-2,-1,1,2) \text{ pcr } 6_q \end{array}\right\}$$

$$P(1,2,3,5)q = \frac{1}{720}\left\{\begin{array}{l} 4q^3+66q^2+324q+451 \\ +45\ (1,-1) \text{ pcr } 2_q \\ +80\ (1,-1,0) \text{ pcr } 3_q \\ +144(1,0,0,0,-1) \text{ pcr } 5_q \end{array}\right\}$$

$$P(1,2,2,3,4)q = \frac{1}{6912}\left\{\begin{array}{l} 6q^4+144q^3+1194q^2+3960q+4267 \\ +(54q^2+648q+1701)(1,-1) \text{ pcr } 2_q \\ +256\ (2,-1,-1) \text{ pcr } 3_q \\ +432\ (1,0,-1,0) \text{ pcr } 4_q \end{array}\right\}$$

$$P(8)q = \frac{1}{8}\left\{\begin{array}{l} 1 \\ +1\ (1,-1) \text{ pcr } 2_q \\ +2\ (1,0,-1,0) \text{ pcr } 4_q \\ +8(1,0,0,0,-1,0,0,0) \text{ pcr } 8_q \end{array}\right\}$$

$$\begin{aligned}P(7, 8)q=\tfrac{1}{112}\Big\{&2q+43\\ &+7 \quad (1, -1) \text{ pcr } 2_q\\ &+14 \quad (1, -1, -1, 1) \text{ pcr } 4_q\\ &+16\,(3, 2, 1, 0, -1, -2, -3) \text{ pcr } 7_q\\ &+56\,(0, -1, -1, 0, 0, 1, 1, 0) \text{ pcr } 8_q\Big\},\end{aligned}$$

which are, I think, worth preserving.

Received April 14,—Read May 3 and 10, 1855.

I proceed to discuss the following problem: "To find in how many ways a number q can be made up as a sum of m terms with the elements 0, 1, 2, ...k, each element being repeatable an indefinite number of times." The required number of partitions is represented by

$$P(0, 1, 2, ..k)^m q,$$

and the number of partitions of q less the number of partitions of $q-1$ is represented by

$$P'(0, 1, 2, ..k)^m q.$$

We have, as is well known,

$$P(0, 1, 2, ..k)^m q = \text{coefficient } x^q z^m \text{ in } \frac{1}{(1-z)(1-xz)..(1-x^k z)},$$

where the expansion is to be effected in ascending powers of z. Now

$$\frac{1}{(1-z)(1-xz)..(1-x^k z)} = 1+\frac{1-x^{k+1}}{1-x}z+\frac{(1-x^{k+1})(1-x^{k+2})}{(1-x)(1-x^2)}z^2+ \&c.,$$

the general term being

$$\frac{(1-x^{k+1})(1-x^{k+2})..(1-x^{k+m})}{(1-x)(1-x^2)..(1-x^m)}z^m,$$

or, what is the same thing,

$$\frac{(1-x^{m+1})(1-x^{m+2})..(1-x^{m+k})}{(1-x)(1-x^2)..(1-x^k)}z^m,$$

and consequently

$$P(0, 1, 2, ..k)^m q = \text{coefficient } x^q \text{ in } \frac{(1-x^{m+1})(1-x^{m+2})..(1-x^{m+k})}{(1-x)(1-x^2)..(1-x^k)};$$

to transform this expression I make use of the equation

$$(1+xz)(1+x^2 z)..(1+x^k z)=1+\frac{x(1-x^k)}{1-x}z+\frac{x^3(1-x^k)(1-x^{k-1})}{(1-x)(1-x^2)}z^2+ \&c.,$$

where the general term is

$$x^{\frac{1}{2}s(s+1)}\frac{(1-x^k)(1-x^{k-1})..(1-x^{k-s+1})}{(1-x)(1-x^2)..(1-x^s)}z^s,$$

and the series is a finite one, the last term being that corresponding to $s=k$, viz. $x^{\frac{1}{2}k(k+1)}z^k$. Writing $-x^m$ for z, and substituting the resulting value of

$$(1-x^{m+1})(1-x^{m+2})\ldots(1-x^{m+k})$$

in the formula for $\text{P}(0, 1, 2, \ldots k)^m q$, we have

$$\text{P}(0, 1, 2, \ldots k)^m q=\Sigma_s\left\{(-)^s \text{ coefficient } x^q \text{ in } \frac{x^{sm+\frac{1}{2}s.s+1}}{(1-x)(1-x^2)\ldots(1-x^s)(1-x)(1-x^2)\ldots(1-x^{k+s})}\right\},$$

where the summation extends from $s=0$ to $s=k$; but if for any value of s between these limits $sm+\frac{1}{2}s(s+1)$ becomes greater than q, then it is clear that the summation need only be extended from $s=0$ to the last preceding value of s, or what is the same thing, from $s=0$ to the greatest value of s, for which $q-sm-\frac{1}{2}s(s+1)$ is positive or zero.

It is obvious, that if $q>km$, then

$$\text{P}(0, 1, 2\ldots k)^m q=0;$$

and moreover, that if $\theta \not< \frac{1}{2}km$, then

$$\text{P}(0, 1, 2, \ldots k)^m \theta=\text{P}(0, 1, 2\ldots k)^m . km-\theta,$$

so that we may always suppose $q \not> \frac{1}{2}km$. I write therefore $q=\frac{1}{2}(km-\alpha)$ where α is zero or a positive integer not greater than km, and is even or odd according as km is even or odd. Substituting this value of q and making a slight change in the form of the result, we have

$$\text{P}(0, 1, 2\ldots k)^m \tfrac{1}{2}(km-\alpha)=\Sigma_s\left\{(-)^s \text{coeff. } x^{(\frac{1}{2}k-s)m} \text{ in } \frac{x^{\frac{1}{2}\alpha+\frac{1}{2}s.s+1}}{(1-x)(1-x^2)\ldots(1-x^s)(1-x)(1-x^2)\ldots(1-x^{k-s})}\right\},$$

where the summation extends from $s=0$ to the greatest value of s, for which $(\frac{1}{2}k-s)m-\frac{1}{2}\alpha-\frac{1}{2}s(s+1)$ is positive or zero. But we may, if we please, consider the summation as extending, when k is even, from $s=0$ to $s=\frac{1}{2}k-1$, and when k is odd, from $s=0$ to $s=\frac{1}{2}(k-1)$, the terms corresponding to values of s greater than the greatest value for which $(\frac{1}{2}k-s)m-\frac{1}{2}\alpha-\frac{1}{2}s(s+1)$ is positive or zero, being of course equal to zero. It may be noticed, that the fraction will be a proper one if $\alpha<(k-s)(k-s+1)$; or substituting for s its greatest value, the fraction will be a proper one for all values of s, if, when k is even, $\alpha<\frac{1}{4}k(k+2)$, and when k is odd, $\alpha<\frac{1}{4}(k+1)(k+3)$.

We have in a similar manner,

$$\text{P}'(0, 1, 2 \ldots k)^m q=\text{coefficient } x^q z^m \text{ in } \frac{1-x}{(1-z)(1-xz)\ldots(1-x^k z)},$$

which leads to

$$\text{P}'(0, 1, 2\ldots k)^m \tfrac{1}{2}(km-\alpha)=\Sigma_s\left\{(-)^s \text{coeff. } x^{(\frac{1}{2}k-s)m} \text{ in } \frac{x^{\frac{1}{2}\alpha+s(s+1)}}{(1-x^2)\ldots(1-x^s)(1-x)(1-x^2)\ldots(1-x^{k-s})}\right\},$$

where the summation extends, as in the former case, from $s=0$ to the greatest value of s, for which $(\frac{1}{2}k-s)m-\frac{1}{2}\alpha-\frac{1}{2}s(s+1)$ is positive or zero, or, if we please, when k is even, from $s=0$ to $s=\frac{1}{2}k-1$, and when s is odd, from $s=0$ to $s=\frac{1}{2}(k-1)$. The condition, in order that the fraction may be a proper one for all values of s, is, when k is even, $\alpha+1<\frac{1}{4}k(k+2)$, and when k is odd, $\alpha+1<\frac{1}{4}(k+1)(k+3)$.

To transform the preceding expressions, I write when k is odd x^2 instead of x, and I put for shortness θ instead of $\frac{1}{2}k-s$ or $2(\frac{1}{2}k-s)$, and γ instead of $\frac{1}{2}\alpha+\frac{1}{2}s(s+1)$ or $\alpha+s(s+1)$; we have to consider an expression of the form

$$\text{coefficient } x^{\theta m} \text{ in } \frac{x^\gamma}{\mathrm{F}x},$$

where $\mathrm{F}x$ is the product of factors of the form $1-x^a$. Suppose that a' is the least common multiple of a and θ, then $(1-x^{a'})\div(1-x^a)$ is an integral function of x, equal χx suppose, and $1\div(1-x^a)=\chi x\div(1-x^{a'})$. Making this change in all the factors of $\mathrm{F}x$ which require it (*i. e.* in all the factors except those in which a is a multiple of θ), the general term becomes

$$\text{coefficient } x^{\theta m} \text{ in } \frac{x^\gamma \mathrm{H}x}{\mathrm{G}x},$$

where $\mathrm{G}x$ is a product of factors of the form $1-x^{a'}$, in which a' is a multiple of θ, *i. e.* $\mathrm{G}x$ is a rational and integral function of x^θ. But in the numerator $x^\gamma\mathrm{H}x$ we may reject, as not contributing to the formation of the coefficient of $x^{\theta m}$, all the terms in which the indices are not multiples of θ; the numerator is thus reduced to a rational and integral function of x^θ, and the general term is therefore of the form

$$\text{coefficient } x^{\theta m} \text{ in } \frac{\lambda(x^\theta)}{\varkappa(x^\theta)},$$

or what is the same thing, of the form

$$\text{coefficient } x^{m} \text{ in } \frac{\lambda x}{\varkappa x}.$$

Where $\varkappa x$ is the product of factors of the form $1-x^a$, and λx is a rational and integral function of x, the particular value of the fraction depends on the value of s; and uniting the different terms, we have an expression

$$\text{coefficient } x^{m} \text{ in } \mathrm{S}_s\,(-)^s \frac{\lambda x}{\varkappa x},$$

which is equivalent to

$$\text{coefficient } x^{m} \text{ in } \frac{\phi x}{fx},$$

where fx is a product of factors of the form $1-x^a$, and ϕx is a rational and integral function of x. And it is clear that the fraction will be a proper one when each of the fractions in the original expression is a proper fraction, *i. e.* in the case of $\mathrm{P}(0,1,2\,..\,k)^m\frac{1}{2}(km-\alpha)$, when for k even $\alpha<\frac{1}{4}k(k+2)$, and for k odd $\alpha<\frac{1}{4}(k+1)(k+3)$; and in the case of $\mathrm{P}'(0,1,2\,..\,k)^m\frac{1}{2}(km-\alpha)$, when for k even $\alpha+1<\frac{1}{4}k(k+2)$, and for k odd $\alpha+1<\frac{1}{4}(k+1)(k+3)$.

We see, therefore, that

$$\mathrm{P}(0,1,2\,..\,k)^m\tfrac{1}{2}(km-\alpha),$$

and

$$\mathrm{P}'(0,1,2\,..\,k)^m\tfrac{1}{2}(km-\alpha),$$

are each of them of the form

$$\text{coefficient } x^{m} \text{ in } \frac{\phi x}{fx},$$

where fx is the product of factors of the form $1-x^a$, and up to certain limiting values of α the fraction is a proper fraction. When the fraction $\frac{\phi x}{fx}$ is known, we may therefore obtain by the method employed in the former part of this Memoir, analytical expressions (involving prime circulators) for the functions P and P′.

As an example, take $\qquad P(0,1,2,3)^m\tfrac{3}{2}m,$
which is equal to

$$\text{coefficient } x^{3m} \text{ in } \frac{1}{(1-x^2)(1-x^4)(1-x^6)}$$

$$-\text{coefficient } x^m \text{ in } \frac{1}{(1-x^2)(1-x^2)(1-x^4)}.$$

The multiplier for the first fraction is

$$\frac{(1-x^6)(1-x^{12})}{(1-x^2)(1-x^4)},$$

which is equal to $\qquad 1+x^2+2x^4+x^6+2x^8+x^{10}+x^{12}.$

Hence, rejecting in the numerator the terms the indices of which are not divisible by 3, the first term becomes

$$\text{coefficient } x^{3m} \text{ in } \frac{1+x^6+x^{12}}{(1-x^6)(1-x^{12})(1-x^6)},$$

or what is the same thing, the first term is

$$\text{coefficient } x^m \text{ in } \frac{1+x^2+x^4}{(1-x^2)^2(1-x^4)};$$

and the second term being

$$-\text{coefficient } x^m \text{ in } \frac{x^2}{(1-x^2)^2(1-x^4)},$$

we have $\qquad P(0,1,2,3)^m\tfrac{3}{2}m = \text{coefficient } x^m \text{ in } \frac{1+x^4}{(1-x^2)^2(1-x^4)}.$

And similarly it may be shown, that

$$P(0,1,2,3)^m\tfrac{1}{2}(3m-1) = \text{coefficient } x^m \text{ in } \frac{x+x^3}{(1-x^2)^2(1-x^4)}.$$

As another example, take $\qquad P'(0,1,2,3,4,5)\tfrac{5}{2}m,$
which is equal to

$$\text{coefficient } x^{5m} \text{ in } \frac{1}{(1-x^4)(1-x^6)(1-x^8)(1-x^{10})}$$

$$-\text{coefficient } x^{3m} \text{ in } \frac{x^2}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)}$$

$$+\text{coefficient } x^m \text{ in } \frac{x^6}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}.$$

The multiplier for the first fraction is

$$\frac{(1-x^{20})(1-x^{30})(1-x^{40})}{(1-x^4)(1-x^6)(1-x^8)},$$

which is a function of x^2 of the order 36, the coefficients of which are

1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 4, 4, 6, 4, 6, 5, 7, 5, 7, 5, 7, 5, 6, 4, 6, 4, 4, 3, 4, 2, 3, 1, 2, 1, 1, 0, 1,

and the first part becomes therefore

$$\text{coefficient } x^m \text{ in } \frac{1+x^2+4x^4+5x^6+7x^8+4x^{10}+3x^{12}}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)}.$$

The multiplier for the second fraction is

$$\frac{(1-x^6)(1-x^{12})(1-x^{24})}{(1-x^2)(1-x^4)(1-x^8)},$$

which is a function of x^2 of the order 14, the coefficients of which are

1, 1, 2, 1, 3, 2, 3, 1, 3, 2, 3, 1, 2, 1, 1;

and the second term becomes

$$-\text{coefficient } x^m \text{ in } \frac{2x^2+2x^4+3x^6+x^8+x^{10}}{(1-x^2)^2(1-x^4)(1-x^8)};$$

and the third term is $\quad \text{coefficient } x^m \text{ in } \dfrac{x^6}{(1-x^2)(1-x^4)^2(1-x^6)}.$

Now the fractions may be reduced to a common denominator

$$(1-x^2)(1-x^4)(1-x^6)(1-x^8)$$

by multiplying the terms of the second fraction by $\frac{1-x^6}{1-x^2}(=1+x^2+x^4)$, and the terms of the third fraction by $\frac{1-x^8}{1-x^4}(=1+x^4)$; performing the operations and adding, the numerator and denominator of the resulting fraction will each of them contain the factor $1-x^2$; and casting this out, we find

$$\mathrm{P}(0,1,2,3,4,5)^m\tfrac{5}{2}m=\text{coefficient } x^m \text{ in } \frac{1-x^6+x^{12}}{(1-x^4)(1-x^6)(1-x^8)}.$$

I have calculated by this method several other particular cases, which are given in my "Second Memoir upon Quantics;" the present researches were in fact made for the sake of their application to that theory.

Received April 20,—Read May 3 and 10, 1855.

Since the preceding portions of the present Memoir were written, Mr. SYLVESTER has communicated to me a remarkable theorem which has led me to the following additional investigations*.

Let $\frac{\phi x}{fx}$ be a rational fraction, and let $(x-x_1)^k$ be a factor of the denominator fx, then if

$$\left\{\frac{\phi x}{fx}\right\}_{x_1}$$

* Mr. SYLVESTER's researches are published in the Quarterly Mathematical Journal, July 1855, and he has there given the general formula as well for the circulating as the non-circulating part of the expression for the number of partitions.—Added 23rd February, 1856.—A. C.

denote the portion which is made up of the simple fractions having powers of $x-x_1$ for their denominators, we have by a known theorem

$$\left\{\frac{\phi x}{fx}\right\}_{x_1} = \text{coefficient } \frac{1}{z} \text{ in } \frac{1}{x-x-z}\,\frac{\phi(x_1+z)}{f(x_1+z)}.$$

Now by a theorem of JACOBI's and CAUCHY's,

$$\text{coefficient } \frac{1}{z} \text{ in } \mathrm{F}z = \text{coefficient } \frac{1}{t} \text{ in } \mathrm{F}(\psi t)\psi' t\,;$$

whence, writing $x_1+z=x_1e^{-t}$, we have

$$\left\{\frac{\phi x}{fx}\right\}_{x_1} = \text{coefficient } \frac{1}{t} \text{ in } \frac{x_1}{x_1-xe^t}\,\frac{\phi(x_1e^{-t})}{f(x_1e^{-t})}.$$

Now putting for a moment $x=x_1e^\theta$, we have

$$\frac{1}{x_1-xe^t}=\frac{1}{x_1(1-e^{\theta+t})}=\frac{1}{x_1(1-e^\theta)}+\partial_\theta\,\frac{1}{x_1(1-e^\theta)}+\dots$$

and $\partial_\theta=x\partial_x$, whence

$$\frac{1}{x_1-xe^t}=\frac{1}{x_1-x}+\frac{t}{1}x\partial_x\frac{1}{x_1-x}+\frac{t^2}{1.2}(x\partial_x)^2\,\frac{1}{x_1-x}+\dots,$$

the general term of which is

$$\frac{t^{s-1}}{\Pi(s-1)}(x\partial_x)^{s-1}\frac{1}{x_1-x}.$$

Hence representing the general term of

$$\frac{x_1\phi(x_1e^{-t})}{f(x_1e^{-t})}$$

by $\chi x_1 t^{-s}$, so that

$$\chi x_1 = \text{coefficient } \frac{1}{t} \text{ in } t^{s-1}\frac{x_1\phi(x_1e^{-t})}{f(x_1e^{-t})},$$

we find, writing down only the general term

$$\left\{\frac{\phi x}{fx}\right\}_{x_1} = \dots+\frac{1}{\Pi(s-1)}\,(x\partial_x)^{s-1}\frac{\chi x_1}{x_1-x}+\dots$$

where the value of χx_1 depends upon that of s, and where s extends from $s=1$ to $s=k$.

Suppose now that the denominator is made up of factors (the same or different) of the form $1-x^m$. And let a be any divisor of one or more of the indices m, and let k be the number of the indices of which a is a divisor. The denominator contains the divisor $[1-x^a]^k$, and consequently if ϱ be any root of the equation $[1-x^a]=0$, the denominator contains the factor $(\varrho-x)^k$. Hence writing ϱ for x_1 and taking the sum with respect to all the roots of the equation $[1-x^a]=0$, we find

$$\left\{\frac{\phi x}{fx}\right\}_{[1-x^a]} = \dots+\frac{1}{\Pi(s-1)}(x\partial_x)^{s-1}\mathrm{S}\,\frac{\chi\varrho}{\varrho-x}+\dots$$

$$=\dots+\frac{1}{\Pi(s-1)}\,(x\partial_x)^{s-1}\,\frac{\theta x}{[1-x^a]}+\dots,$$

where $$\chi\varrho = \text{coefficient } \frac{1}{t} \text{ in } t^{s-1}\,\frac{\varrho\phi(\varrho e^{-t})}{f(\varrho e^{-t})},$$

and as before s extends from $s=1$ to $s=k$. We have thus the actual value of the function θx made use of in the memoir.

A preceding formula gives

$$\left\{\frac{\phi x}{fx}\right\}_1 = \text{coefficient } \frac{1}{t} \text{ in } \frac{1}{1-xe^t}\frac{\phi(e^{-t})}{f(e^{-t})},$$

which is a very simple expression for the non-circulating part of the fraction $\frac{\phi x}{fx}$. This is, in fact, Mr. SYLVESTER's theorem above referred to.